Indian Statistical Institute, Bangalore

B. Math (Hons.) Second Year

Second Semester - Ordinary Differential Equations

Backpaper (Final exam) Date: Jun 04, 2025

Maximum Marks: 50 Duration: 3 hours

ANSWER ALL QUESTIONS. EACH QUESTION CARRIES 5 MARKS

(1) Show that the `initial value problem` $y' = xy^2$ on the rectangle $\mathcal{R} := \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}$ has unique solution but the solution is not unique in the domain $S = \{|x| \leq 1, |y| < \infty\}$.

(2) Let $a \in \mathbb{C}$ be such that the real part $\Re(a)$ is nonzero, and let b_1, b_2 be two complex-valued continuous functions on $0 \leq x < \infty$ such that $|b_1(x) - b_2(x)| \leq K$, $(0 \leq x < \infty)$ for some constant $K > 0$. Let φ and ψ be two complex-valued functions that satisfy the equations $y' + ay = b_1(x)$ and $y' + ay = b_2(x)$, respectively. Assume that $\varphi(0) = \psi(0)$. Show that

$$|\varphi(x) - \psi(x)| \leq \frac{K}{\Re(a)} \left[1 - e^{-\Re(a)x}\right]$$

for $0 \leq x < \infty$.

(3) Show that the equation $ydx + (2x - ye^y)dy = 0$ is not exact. Find the integrating factor and solve it.

(4) Solve the following system

$$\begin{cases} \frac{dx}{dt} = 3x + 4y \\ \frac{dy}{dt} = 5x + 6y. \end{cases}$$

(5) Consider Bessel's equation $x^2y'' + xy' + (x^2 - p^2)y = 0$ whose normal form is

$$u'' + \left(1 + \frac{1 - 4p^2}{4x^2}\right) u = 0,$$

where $x > 0$ and $p \geq 0$ and compare with the equation $v'' + v = 0$, then what can you conclude about the zeros of the solution u of the Bessel's equation in each of the following cases: $0 \leq p < 1/2$, $p = 1/2$, and $p > 1/2$.

(6) Define stable and asymptotically stable critical point of the system

$$\begin{cases} \frac{dx}{dt} = F(x, y) \\ \frac{dy}{dt} = G(x, y). \end{cases}$$

Show that the $(0, 0)$ is the spiral simple critical point of the system

$$\begin{cases} \frac{dx}{dt} = -2x + 3y + xy \\ \frac{dy}{dt} = -x + y - 2xy^2. \end{cases}$$

Discuss the stability by using Poincaré theorem.

(7) Show that any non-trivial solution $u(x)$ of $u'' + q(x)u = 0$, $q(x) < 0$ for all x, has at most one zero.

(8) Prove the following: Suppose that φ_1 and φ_2 be a fundamental pair of solutions (and, hence are linearly independent) of $y'' + q(x)y = 0$. If x_1 and x_2 be two consecutive zeros of φ_1, then φ_2 has exactly one zero in (x_1, x_2).

(9) By method of variation of parameters find the complete solution of the following equation

$$y'' + 2y' + y = e^{-x} \log x.$$

(10) Find the general solution the following equation by using *method of Frobenius*

$$2x^2y'' + x(2x+1)y' - y = 0$$

by finding two independent Frobenius series solution.

Good luck!!